Mechanism for Square Trajectory

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ABSTRACT

This paper presents the conceptual-kinematic design of a novel mechanism which performs an exact square trajectory, using only one source of rotational input and without any constraints applied to the output point of motion.
It consists of three mechanical entities merged into an unitary mechanism: a four-bar linkage with parallel and equal cranks, a planetary gear arrangement and a periodic cam with two conjugated surfaces. They all comply with the same central axis of rotation which also defines the fixed frame reference as well as the input reference.
Two proof-of-concept prototypes have been constructed so far and a dynamically optimized design has been simulated in SolidWorks for a self-configurable modular system as an application proposal.
The method of synthesis is presented as well as the analytical approach for cam profile generation.

INTRODUCTION

The mechanism described in this paper can be placed in the category of the exact-straight line mechanisms, exhibiting the additional feature of performing exact 90^{0} angles. This may sounds like a geometrical tautology but it is justified regarding the continuity of the internal rotational motions. There are neither interruptions nor changes in the directions of the rotating elements. They all have full 360^{0} range of rotation and over, as the cycle repeats. Hence, the output point of motion traces the square path continuously and makes instantaneous transition from one side of the square to another.

Figure 1 shows a simplified model for the purpose of clarity and ease of understanding of the working principle. An actual dynamically optimized design is presented in the last chapter and it has a rather complex architecture since it has to deal with torques, load forces, structural consistency and precision.

At this point, three main features of the design can be stated as follows:

- The entire mechanical assembly is exclusively contained inside the square perimeter of the trajectory.
- There is only one source of rotational input.
- The output point of motion traces the square path without the use of any kind of guidance.

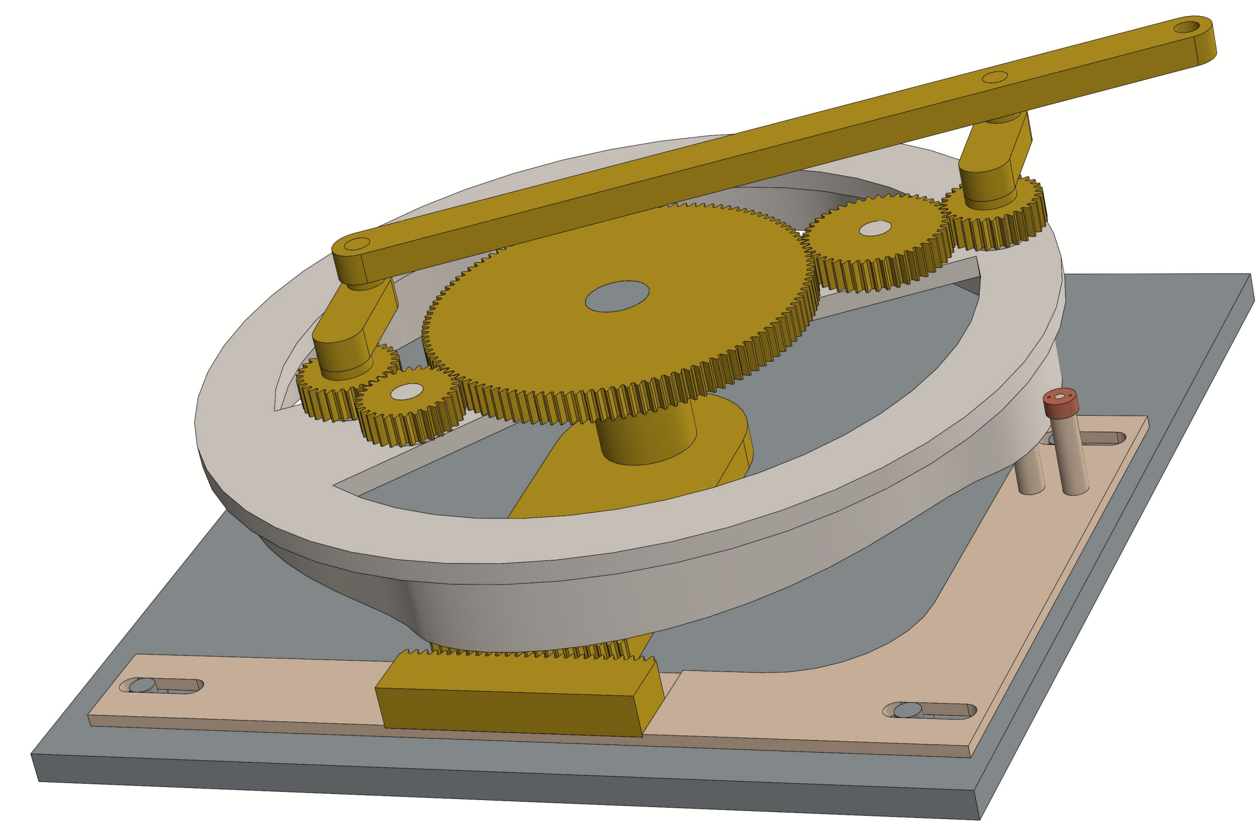

Figure 1. CAD illustration of a mechanism for square trajectory

BRIEF DESCRIPTION OF THE ASSEMBLY ARCHITECTURE

As mentioned in the abstract chapter, there are three mechanical entities involved: a four-bar linkage, a planetary gear arrangement and a periodic cam with two conjugated surfaces. The merging between them is achieved by assigning more than one functional role to certain elements. The planet carrier, besides the obvious role of carrying the planet gears, also plays two additional functions: it represents the frame link for the four-bar linkage configuration and in the same time, it forms a rigid body with the cam profile. In consequence, it also operates the cam rotation. Furthermore, these two combined yield for the planet carrier the master role of input driving rotation for the entire assembly.

The four-bar linkage has a parallelogram configuration and both front and rear cranks are driven synchronously by the planetary gear arrangement. The floating link has an extension where the output point of motion is located at a specific distance from the revolute joint point of the front crank.

The planetary gear system comprises two pairs of planets arranged in chain in order to fulfill the desired direction of cranks rotation, which has to be in opposition to the direction of planet carrier rotation.

The gear ratio between end-planets and sun gear is 4 to 1. Only the front pair of planets do have specific positioning relevance and they are derived from the two kinematic conditions needed to be obeyed. One is the distance between central axis and last planet gear axis, which will provide a parameter for dimensional synthesis. The other is the relative position towards a certain 0 value initialization point on the cam profile.

The rear pair of planets can be located anywhere, given the appropriate change of the floating link shape and considerations regarding dynamics.

The square base and the central axle represents the fixed frame. The sun gear is free to rotate in respect to the central axle and it is constrained only by the follower body through a rack and pinion engagement. The pinion gear sector forms a rigid body with the sun gear. The cam profile has a concave-convex shape which in consequence requires the follower to be equipped with two rollers, designated for the two opposing conjugated surfaces of the cam profile. The follower has a radial position relative to the center of rotation and exhibits oscillatory translational displacement.

MOTION COMPOSITION AND BEHAVIOR

As mentioned above, the rotation of the planet carrier represents the input driving source for the entire assembly. For simplicity, the actual motor engagement is not depicted in figures and illustrations.

Therefore, as the planet carrier revolves continuously, the end-planets will revolve continuously and consequently the cranks will do the same. The bar linkage can be viewed as an extendable arm, by exclusive means of fully rotational linkage elements, which in this case are the two cranks. The configuration of the bar linkage has to be initialized for correct operation since it has an asymmetrical position towards the center of the square. This can be done in either the two extreme positions, which are when the cranks are co-linear with the floating link. When fully folded in one direction, the output point of motion matches the corners of the square path and when folded in the opposite direction, the positions correspond to the middle of the sides.

The latter proves to be more convenient since the cam profile position has to be also matched with a zero value of the follower displacement. This is due to the rack-and-pinion connection between the sun gear and follower body, which closes up a loop chain of engagements. This aspect is somehow counter intuitive but it will be clearer in the next chapters where the rotational mechanical elements are geometrically represented as rotating vectors, parallel lines have conspicuous meaning and the cam profile is mathematically determined.

In conclusion, the cam rotation imparts an oscillatory linear displacement to the follower body, which in turn imposes a circular oscillation of the sun gear. The actual end effect required is the angular displacement transmitted to the cranks. More adequately said, the cranks rotate with variable angular velocity and simultaneously satisfy harmonic and in-phase periodicity relation with planet carrier rotation. Due to this behavior inherited from the closed loop chain of engagements, the output point of motion is kept in coincidence with the entire length of the square sides. Otherwise, the bar-linkage configuration alone provides a perfect match only with the vertices points of the square and the middle points of the sides. In such case, the path generated would be the undesirable "pillow" shaped distortion of a square. This is where the involvement of the cam function provides the exact square trajectory.

SYNTHESIS, ANALYSIS AND MATHEMATICAL DETERMINATION

The underlying geometrical problem illustrated in Figure 2 provides a straightforward solution for path generation synthesis of the bar linkage. The frame link, which in this case is represented by the planet carrier body, actually rotates and therefore creates a geometrical condition for which an unique function can be determined, binding the relation between any arbitrary point position on the square path with a corresponding angle of the planet carrier rotation. Both length dimensions determination and cam profile generation are integrated in the same task.

For this purpose, rotating vectors are designated to mechanical elements.

Vector V_1 and V_2 are rotating vectors and V_3 is a floating vector which has to obey the parallelism condition with vector V_1. Second condition is the opposition between the directions of rotation of vectors V_1 and V_2. This is because Vector V_3 is designated to the extension of the floating link in the bar linkage, vector V_2 is designated to the front crank and vector V_1 represents the rotation of the frame link or planet carrier body.

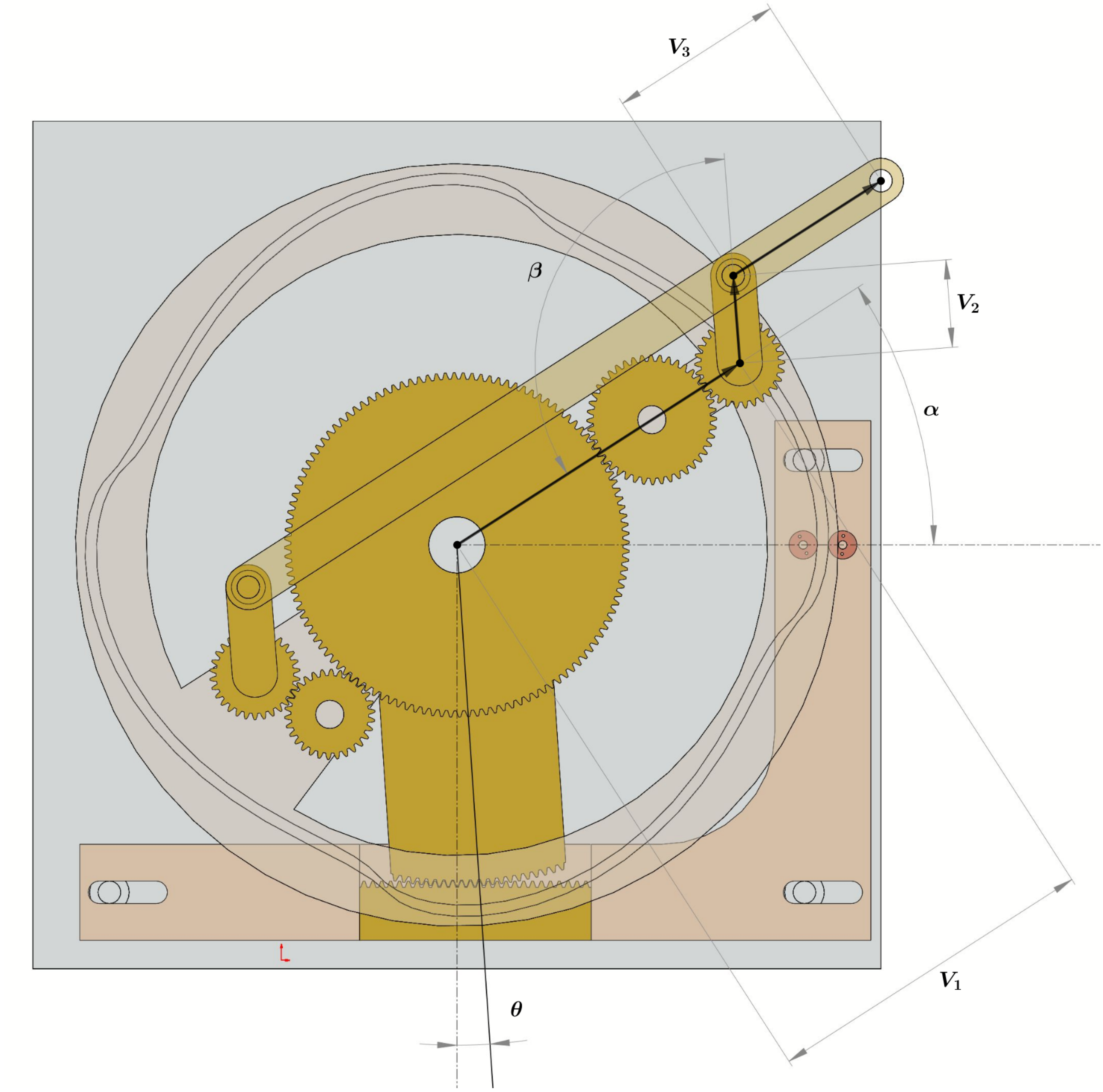

Figure 2. Vector assignment for mechanical architecture

In Figure 3 they are represented in three different states: when fully extended, meaning they are co-linear, point the same direction and are coincident to the diagonal line connecting the center with a corner; when fully folded and coincident with a bisector of a side of the square; and finally, for an arbitrary intermediate position, bearing in mind the two conditions stated above: parallelism between V_1 and V_3 and opposition of directions of rotation for V_1 and V_2.

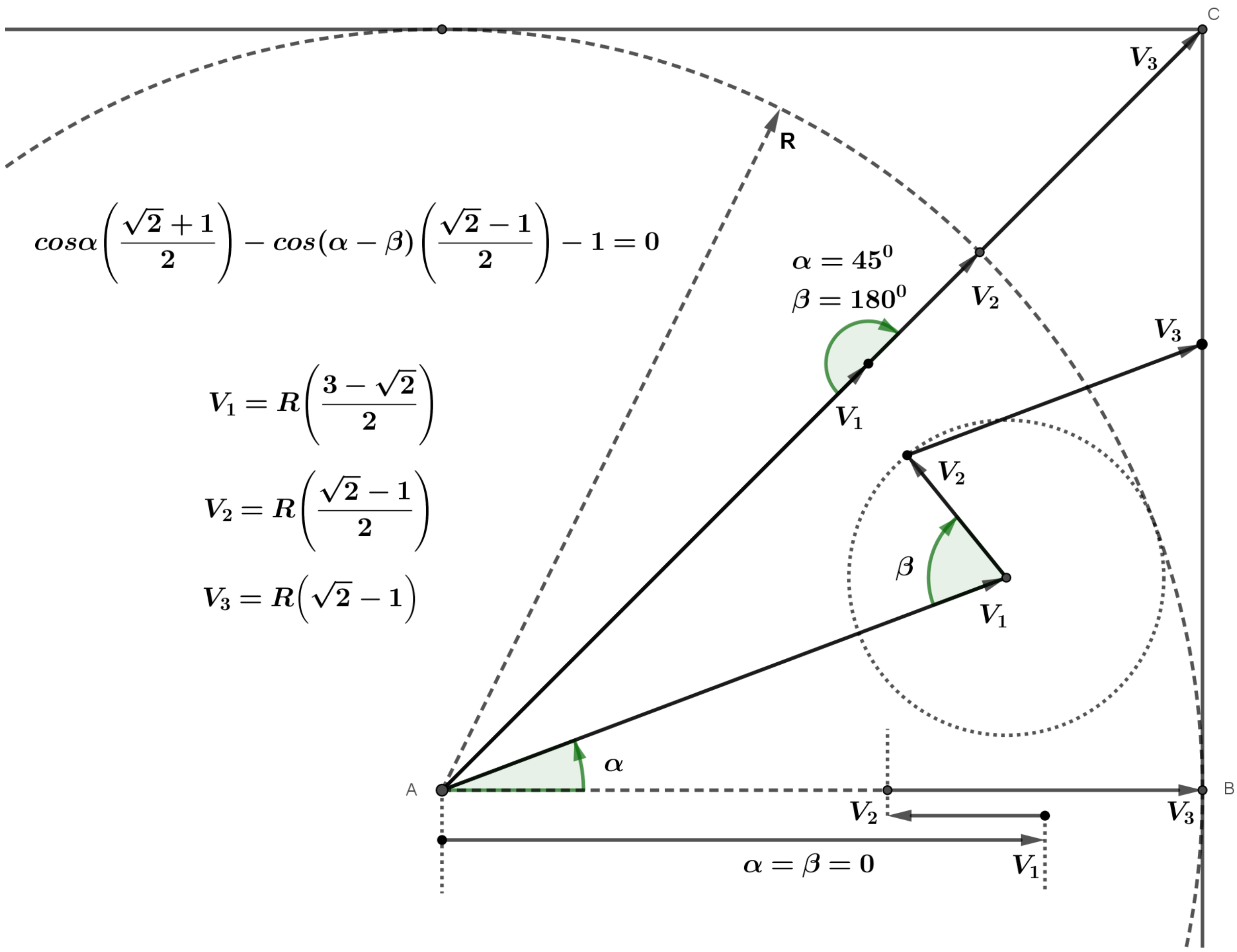

Figure 3. Vector construction for three different positions

In order to determine the lengths or magnitudes, the circle inscribed in the square provides the kind of solution hidden in plain sight. And that is, dividing the hypotenuse of the right triangle ABC into three segments and assign it the fully extended position. One point of division is the intersection between the hypotenuse and the inscribed circle providing the length of V_3 . The second point divides V_1 from V_2 and yields from calculation.

Considering R to be the radius of the inscribed circle and having a value of choice, the right triangle ABC provides all necessary data.

Combining the addition and subtraction of the magnitudes, from the fully folded and fully extended positions and then plugging them to Pythagorean theorem, yields the lengths of all vectors. If:

$$V_1+V_2+V_3=R\sqrt{2} \tag{1}$$

$$V_1-V_2+V_3=R \tag{2}$$

$$V_1+V_2=R \tag{3}$$

then,

$$V_3=R(\sqrt{2}-1) \tag{4}$$

$$V_2=R\left(\frac{\sqrt{2}-1}{2}\right) \tag{5}$$

$$V_1=R\left(\frac{3-\sqrt{2}}{2}\right) \tag{6}$$

Next step is to focus on the arbitrary intermediate position of the vector construction and withdraw the idea of a continuum between the two extreme positions. From this, it can be easily concluded that, while vector V_1 performs a 45^{o} rotation, vector V_2 has to perform 180^{o} rotation in order to match the two extremes. Obviously they have an in-phase periodicity ratio of 4. This ratio contradicts the instantaneous relationship required between angles **α** and **β** for an arbitrary position where the condition of coincidence between the tip of vector V_3 and the side of the square has to be fulfilled. The actual relation between angles **α** and **β**, that can simultaneously satisfy both the harmonic periodicity ratio for the extreme positions and the instantaneous ratio of an arbitrary intermediate position, is yet to be determined.

This task involves a quite large arsenal of trigonometric tools. Drawing additional lines between intersection points upon vector construction in Figure 3 would reveal solvable geometry. Combining the laws of sine, cosine and tangent and then going through a tedious process of plugging trigonometric identities, equation (7) can be found where all scalar terms are naturally reduced. The relation between **α** angle and **β** angle is expressed only in terms of cosine functions and constant coefficients related to square root of 2.

$$\cos\alpha\left(\frac{\sqrt{2}+1}{2}\right)-\cos(\alpha-\beta)\left(\frac{\sqrt{2}-1}{2}\right)-1=0 \tag{7}$$

Geometrical analysis of only one half-side of the square path is sufficient, due to reflection symmetry of the vector construction related to diagonal lines. The transition through the corners is performed continuously and in perfect 90^0 angles, preserving the directions of vectors rotation. The importance of this feature is crucial for operating a chain connection of several mechanisms. Last chapter describes such application proposal.

Returning to mechanical configuration, inverse kinematics can be applied in order to obtain the pitch curve of the cam profile. Hence, **β** angle describes the cranks rotation in respect to the frame link and and **α** angle describes the planet carrier rotation, bearing in mind that the planet carrier identifies with the frame link. Since the two cranks are driven by the planetary gear arrangement, same mathematical relations are valid for the gears. The angular displacement of the cranks reflect the angular displacement of the corresponding planet gears. Due to periodicity relation between V_2 and V_1, the angular displacement required for a planet gear is:

$$\gamma = 4\alpha - \beta \tag{8}$$

In consequence, the angular displacement of the sun gear will be:

$$\theta = \alpha - \left(\frac{\beta}{4}\right) \tag{9}$$

And since the pinion sector gear forms a rigid body with the sun gear, proceeding further through the rack-and-pinion engagement, the linear displacement of the follower can be found:

$$S = \frac{\left(\alpha - \frac{\beta}{4}\right)\pi r}{180^0} \tag{10}$$

where r is the radius of the pinion gear and has a value of choice.

Final procedure is to parameterize the equation (7) by targeting **α** angle as an independent variable. Therefore, an instantaneous discrete value of the follower displacement **S** can be found for any given value of **α** angle. Due to the central symmetry of assembly architecture, these two values, **α** and **S**, can be used as polar coordinates for graphical generation of the cam profile. Figure 4 shows one pitch curve in relation to a chosen pitch circle.

There is a direct visible symmetry of the pitch curve and it belongs to the global π/2 periodicity. There is also a less conspicuous symmetry and has a algebraic nature. This regards the instantaneous radius value of the pitch curve in relation with intersection points between pitch circle and pitch curve. Intersection points correspond to 0 value displacement of the follower.

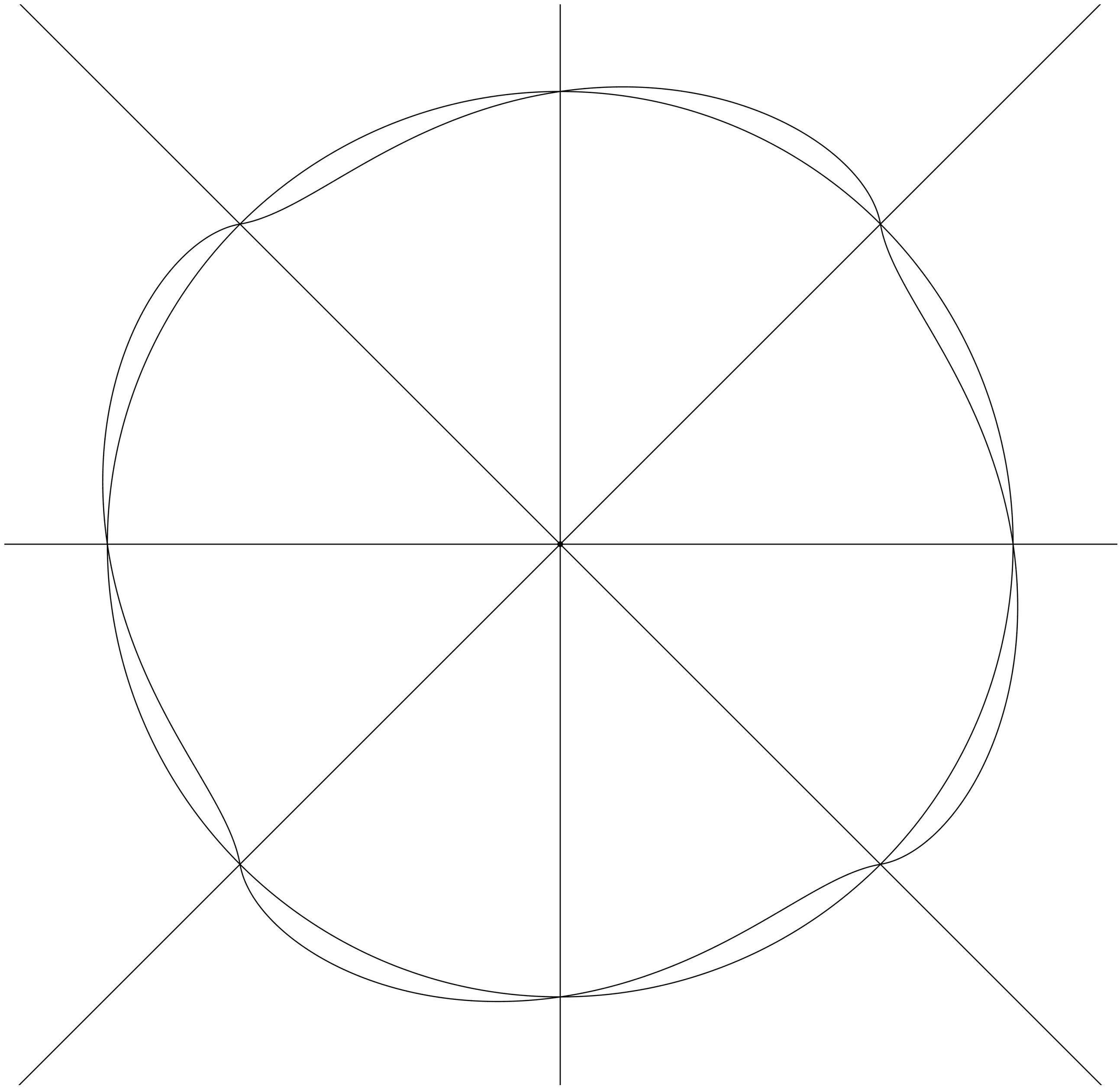

Figure 4. Pitch curve of the cam profile in relation to pitch circle

As it can be observed, any chosen $\pi/2$ sector is bounded by intersection points and also contain one which splits that particular sector in half. One region remains outside the pitch circle and another one inside. The symmetry resides in identical deviation value for any two equally spaced points from an intersection point in opposite directions, one being located on the outer region of the pitch curve and the other one on the inner region. The deviation is measured radially, outwards and inwards from the radius of the pitch circle.

In other words, two equally spaced rays from an intersection point would have their instantaneous values as (q+S) or (q-S), given q the radius of the pitch circle and S being the instantaneous follower displacement. This symmetry would be visually obvious if the function would be mapped in cartesian coordinates, having **α** angle replaced with a constant incremental argument. The intersection points would become symmetry points. But because the function is wrapped around a circle in polar coordinates, the geometrical symmetry is broken and only the algebraic one is preserved. This dissociation explains the velocity function of the follower behavior. It exhibits dimensional symmetry of the displacement regarding maximum left and right stroke positions while velocity varies asymmetrically. Proceeding through the rack and pinion engagement, the sun gear oscillates geometrically like a pendulum, with a central zero displacement but with variable velocity asymmetrically distributed. That is why the inflection points have asymmetric positions on the pitch curve. The pitch curves obtained are smooth and do not exhibit singular points.

The cam profile is constructed by radially scaling the two pitch curves and then regenerating the two conjugated working surfaces by tangency to follower rollers.

Figure 5 illustrates the two conjugated working surfaces and several key positions of the follower rollers.

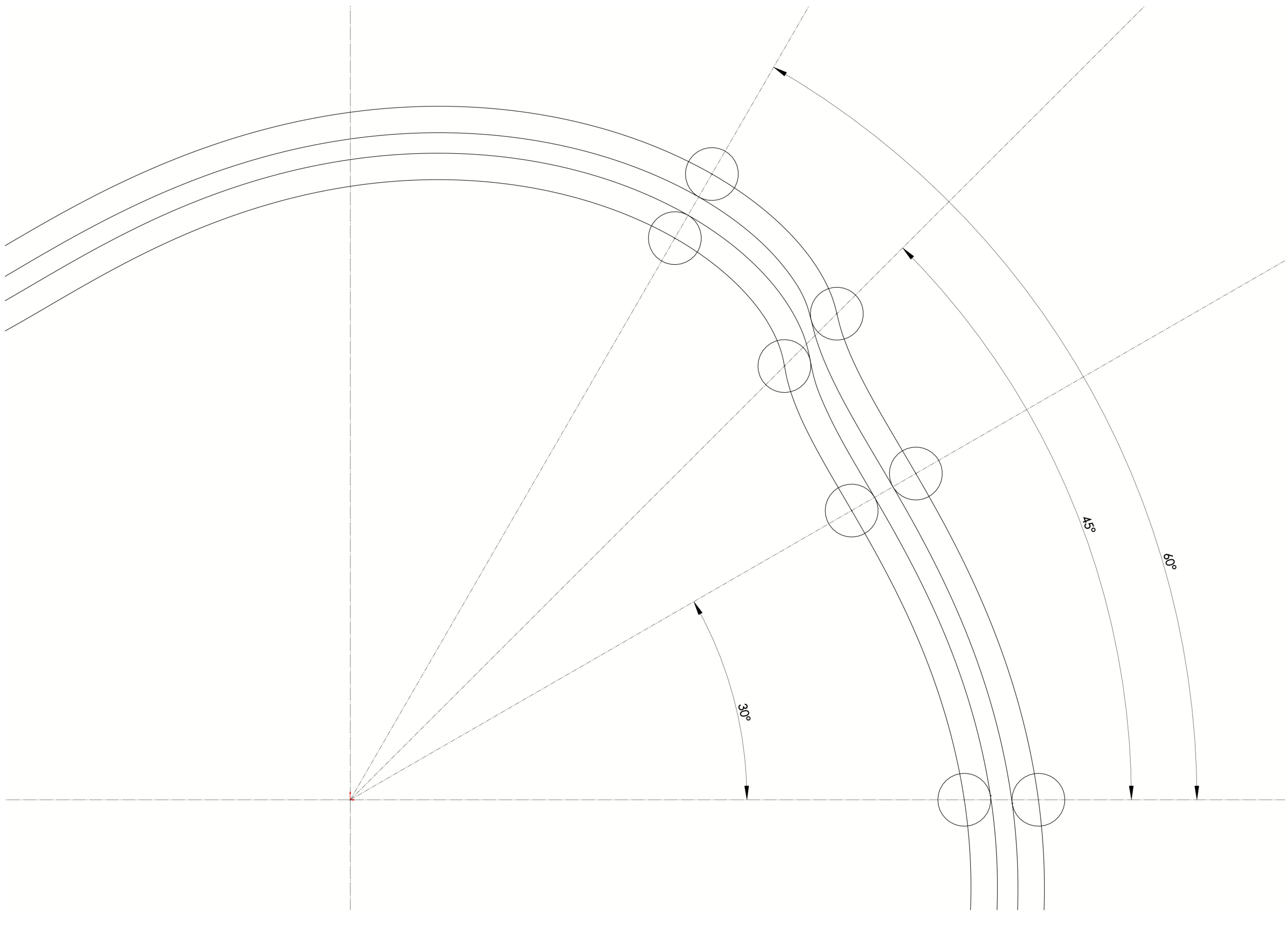

Figure 5. Section through the cam profile

Dimensional choices of certain mechanical components influence the cam profile generation. Thus, the radius of the pinion sector gear determines the amplitude of pitch curve deviation from the pitch circle. Choosing the diameter of the pitch circle itself influences the resolution of the cam.

An interesting aspect to be mentioned is the locations of the minimum and maximum values for the instantaneous radius of the pitch curve. They are located exactly at 30^o and 60^o in respect to initial position and at corresponding values for the rest of the quadrants. In the same time, the zero value displacement positions are located at 0^o, 45^o and 90^o for each quadrant.

APPLICATION PROPOSAL

The first occurring implication of the feature of a constraint free output point of motion is the possibility of a mirror interconnection of two similar mechanisms sharing a common floating link. Figure 6 shows such configuration.

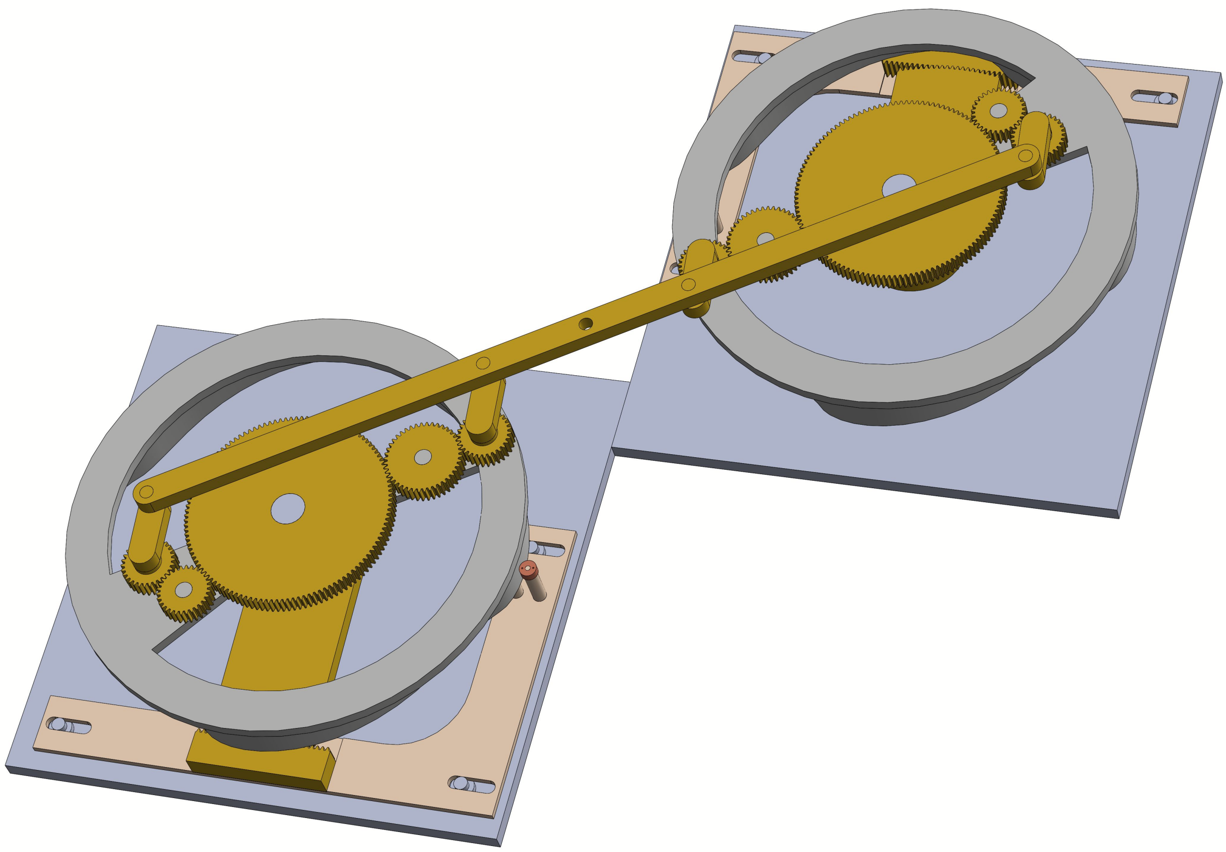

Figure 6. Mirror connection of two identical mechanisms sharing the same floating link

Operated synchronously ,the two mechanisms produce a “orbiting” relative motion between the two square base frames, given the fact that none is considered stationary and the other mobile. In plain words, they go around each other.

The most important characteristic of this behavior is that the square base frames preserve orientation over the full 360^{0} revolution, while maintaining the coincidence between either two sides or two vertices alternatively. Due to this, the square base frames will oppose different sides within each quarter of the cycle while the opposition reference for the planet carriers, and implicitly for the whole internal architecture, does not change. Planet carriers always face each other the initial rotating states. This can be visualized in Figure 7, where rotating vectors describe mechanical architecture as in previous chapters. Reflection symmetry governs yet again the geometry of connection between two mechanisms. It is a double reflection in two orthogonal planes, both containing the common output point of motion, or a 180^{0} rotation by geometric transformation equivalence.

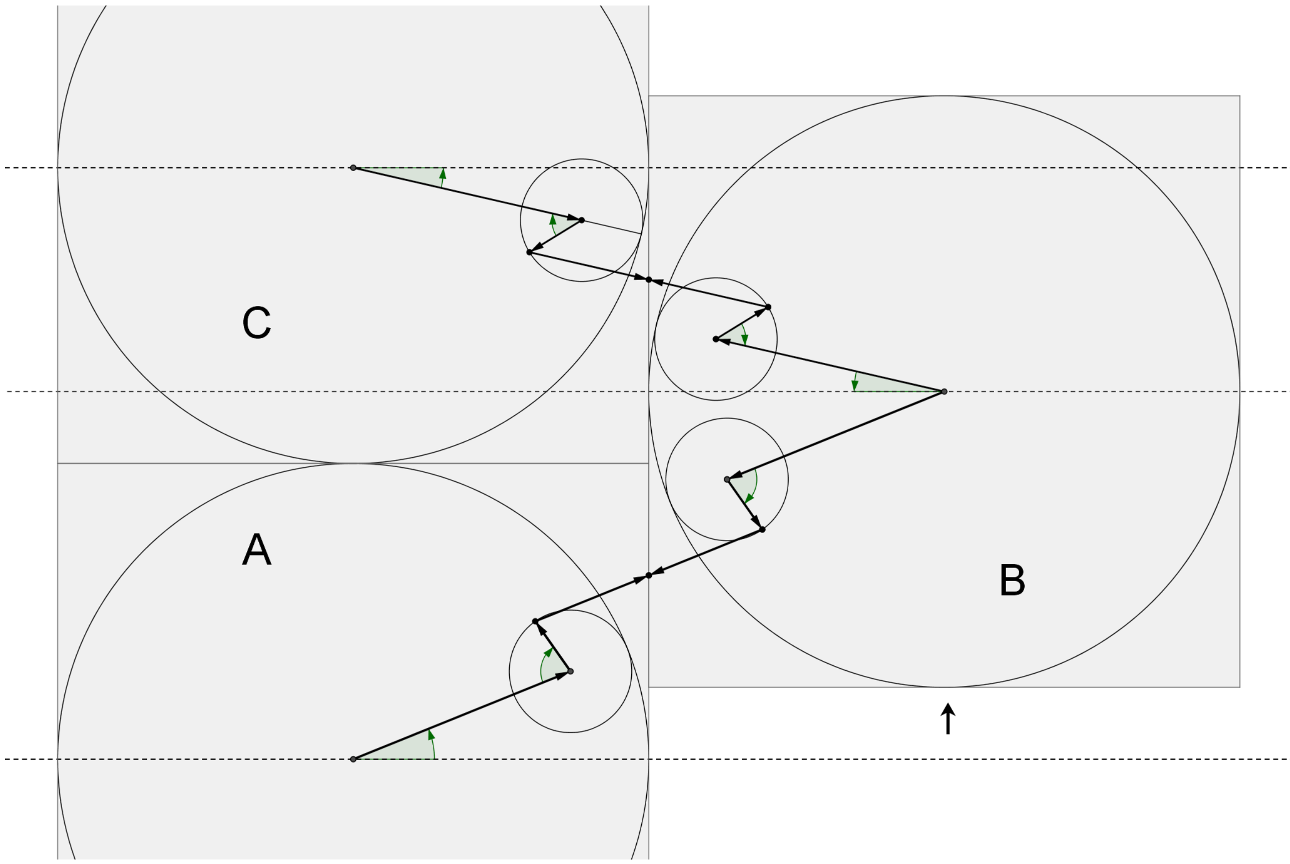

Figure 7. Vector construction for chain-connected mechanisms

For a real application, the square base frame has to be converted into a rigid body comprising two opposing faces of a cube connected by a central axle. This construction becomes a modular entity for a chain connection of multiple mechanisms.

Figure 8 and Figure 9 contain CAD illustrations of a dynamically optimized design for chain-connected operation. It consists of a pair of modules with vacant spaces along the central axles where additional identical mechanisms can be accommodated.
In consequence, each module would posses two separate mechanisms in order to connect with other modules positioned in rear and forward adjacent positions.. The faces comprise a rail-and-roller configuration arranged in diagonal alternation for frictionless operation.

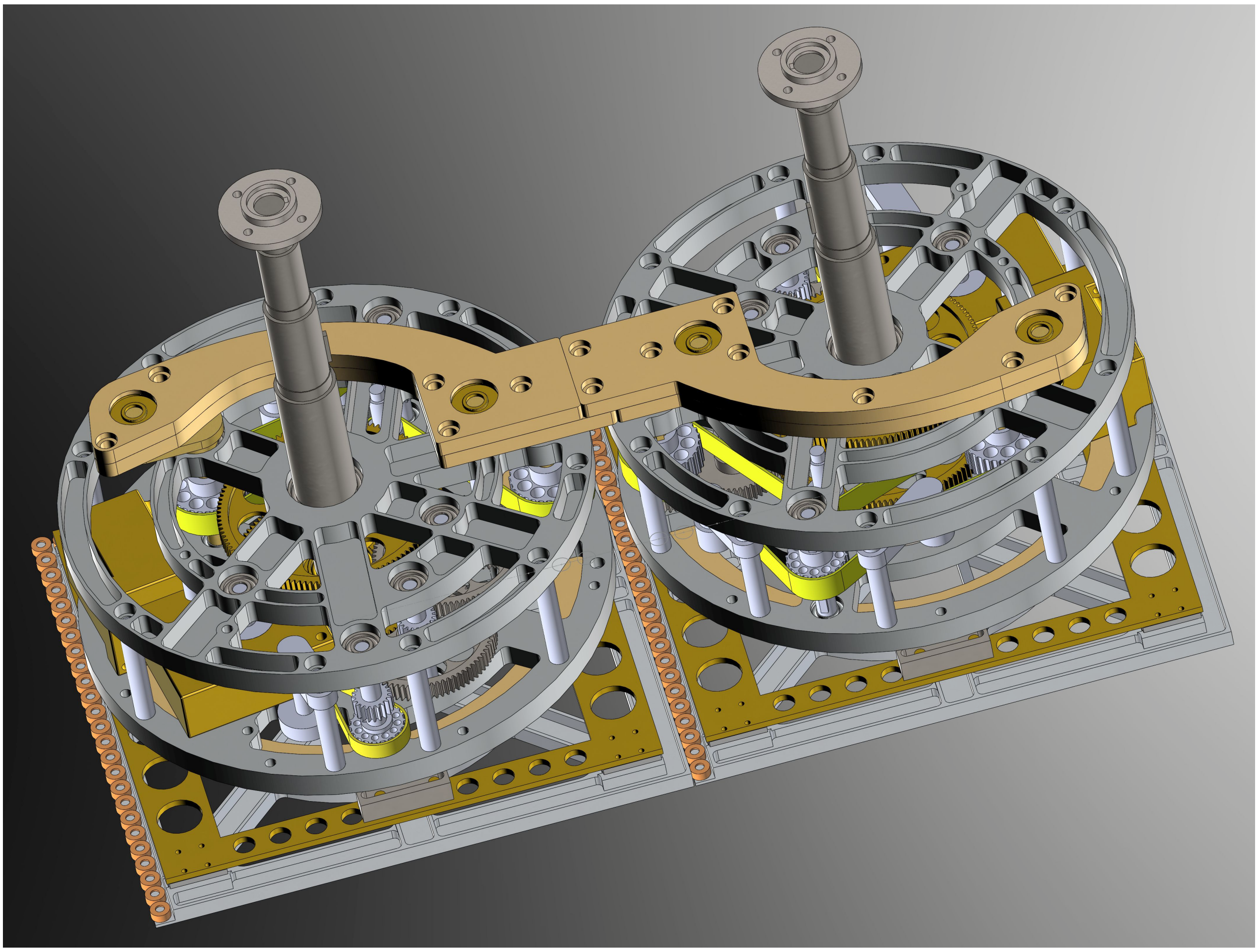

Figure 8. CAD illustration of a dynamically optimized design

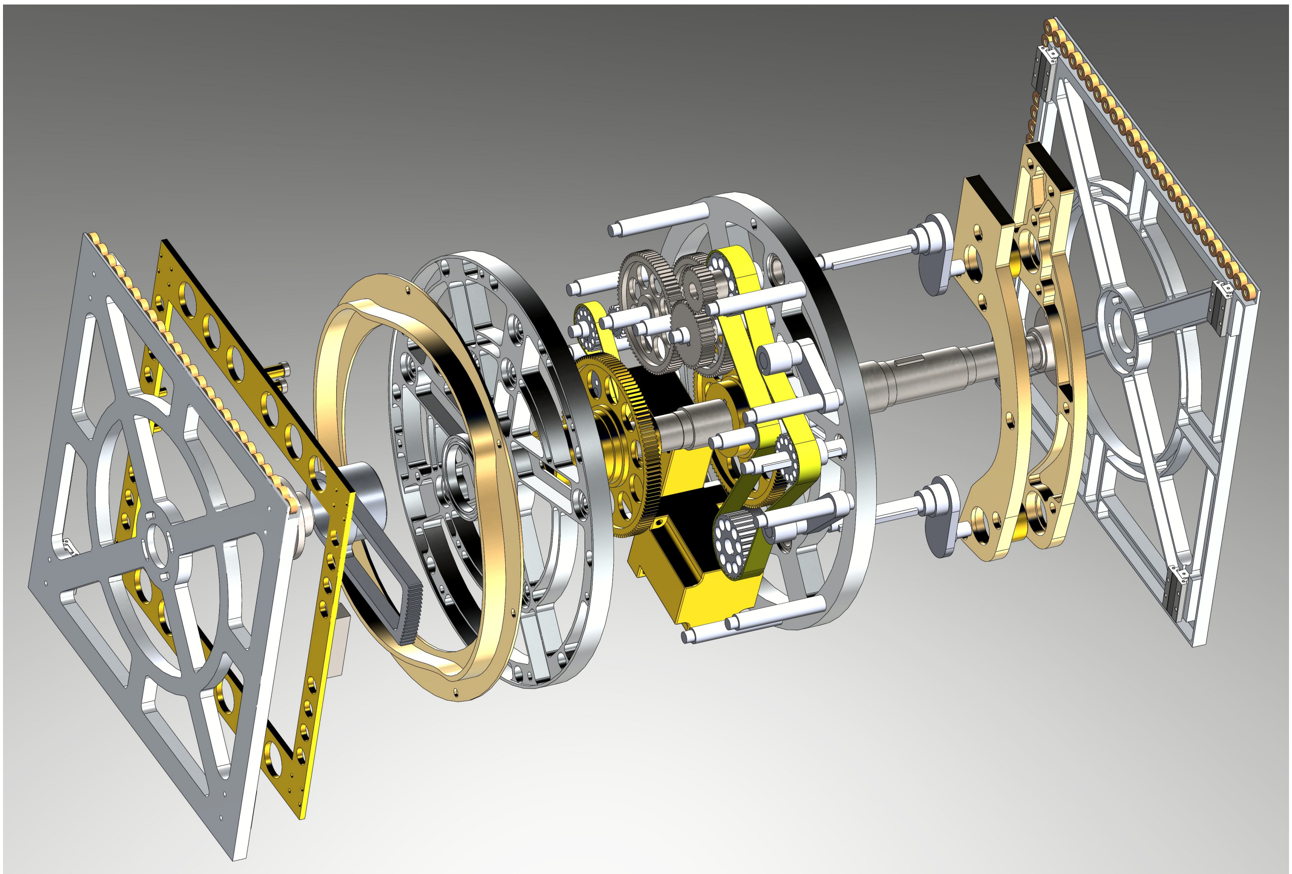

Figure 9. Exploded view of one mechanism

Illustrations in Figure 10 are extracted from an animated simulation of a chain operation of several modules. The initial position is horizontal and by following a certain algorithm of permutations, they self-reconfigure in vertical position. The algorithm has to obey certain rules required for the architectural validity of the self-configurable column. One of them is the restriction of positioning only one module in console-state at a time. Otherwise the structure would collapse. Another one is the requirement of simultaneous operation of two separate connection in a conjugated manner. This is because the initial consecutive order of connections has to remain intact. Horizontal permutation are meant to transfer the weight, that is building up on the vertical formation, onto a single gravitational alignment. It is performed in alternation with vertical operations. Vertical operations require the synchronization of two consecutive connections, while horizontal permutation requires two non-consecutive connections. Understanding the permutation algorithm is difficult without animated simulation. (https://www.raduapetroaia.com/orbit-01)

Figure 10. Snapshots from animated simulation of a self-configurable column

The absolute precision needed for the square path, with exact straight lines and exact 90 degree angles, derives from the above mentioned necessity of simultaneous operation of two separate connections. One of them initiate motion being in side-to-side coincidence and the other one being in corner-to-corner coincidence. That is why approximations are not permitted and a perfect square trajectory is required from each individual mechanism involved.

Video footage with proof-of-concept prototypes:

https://www.raduapetroaia.com/orbit-01

https://www.raduapetroaia.com/mp01